111B Section Week 6

Overview: Work on the following problems one at a time, either by yourself or in small-groups. After a sufficient amount of time has passed, we will discuss the solutions as a class. Attending section counts toward your participation grade.

1. Using the Fundamental Theorem of Ring Homomorphisms, determine all possible unital ring homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$.

2. Let R be a ring. Recall that if $e \in R$ is a central idempotent, then the set $Re = \{re : r \in R\}$ is a subring of R with multiplicative identity equal to e.

   (a) Show that if $e \in R$ is a central idempotent, then Re is a two-sided ideal of R.

   (b) Suppose further that R is a ring with 1 and $e, f \in R$ are central idempotent satisfying $ef = 0 = fe$ and $e + f = 1$. Use the Fundamental Theorem of Homomorphisms to find a unital ring isomorphism $R/Re \to Rf$.

   (c) Let $R = \mathbb{Z}/6\mathbb{Z}$. Find the two nontrivial central idempotents $e, f \in R$ and show that they are orthogonal. Find the isomorphism type of R/Re using part (b).

3. Let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ denote the ring of Gaussian integers.

   (a) Define $(1 + x^2) := \{p(x)(1 + x^2) \in \mathbb{Z}[x] : p(x) \in \mathbb{Z}[x]\}$. Convince yourself that $(1 + x^2)$ is an ideal in $\mathbb{Z}[x]$.

   (b) Use the Fundamental Theorem of Homomorphisms to prove that $\mathbb{Z}[x]/(x^2 + 1) = \mathbb{Z}[i]$.